NON-COMMUTATIVE A-G MEAN INEQUALITY

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Abstract. In this paper we consider non-commutative analogue for the arithmetic-geometric mean inequality

$$a^r b^{1-r} + (r-1)b \geq ra$$

for two positive numbers a, b and $r > 1$. We show that under some assumptions the non-commutative analogue for $a^r b^{1-r}$ which satisfies this inequality is unique and equal to r-mean. The case $0 < r < 1$ is also considered. In particular, we give a new characterization of the geometric mean.

1. Introduction

For any two positive numbers a, b and $r > 1$, we have the arithmetic-geometric mean inequality

$$a^r b^{1-r} + (r-1)b \geq ra.$$

In this paper we consider the non-commutative analogue of this inequality for bounded linear operators on a Hilbert space. In particular, we give a new characterization of the geometric mean. Recently their ingenious paper [5], Carlen and Lieb used a certain non-commutative analogue of this inequality. Their paper is a motivation of our considerations.

There is one obvious non-commutative analogue as follows. For a bounded positive operator X on a Hilbert space, we always have

$$X^r + (r-1) \geq rX.$$

For any two positive invertible operators A, B, set $X = B^{-1/2}AB^{-1/2}$. Then we have

$$(B^{-1/2}AB^{-1/2})^r + (r-1) \geq rB^{-1/2}AB^{-1/2}$$

and hence

$$B^{1/2}(B^{-1/2}AB^{-1/2})^r B^{1/2} + (r-1)B \geq rA.$$

Thus if we consider $B^{1/2}(B^{-1/2}AB^{-1/2})^r B^{1/2}$ (so-called r-mean) as non commutative analogue for $a^r b^{1-r}$, we get a desired inequality.

We conjecture that there is no other example of non-commutative analogue for the above arithmetic-geometric mean inequality. The main result of this paper is

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as follows. We consider "non-commutative analogue" $M(A,B)$ for $a^r b^{1-r}$. More precisely M is a two variable map and $M(A,B)$ is a positive invertible operator for any two positive invertible operators A, B. We assume that

(i) $M(tA,B) = t^r M(A,B)$ for any positive number t,
(ii) $M(A,B)^{-1} = M(A^{-1}, B^{-1})$.

For example,

$$A^{r/2} B^{1-r} A^{r/2}, \qquad B^{(1+2r)/2}(B^6 A^{-2} B^6)^{-r/2} B^{(1+2r)/2}$$

satisfy these conditions. Under these assumptions, if the inequality

$$M(A,B) \geq rA + (1-r)B$$

holds, then we will show that

$$M(A,B) = B^{1/2}(B^{-1/2} A B^{-1/2})^r B^{1/2}.$$

Therefore in a certain sense our conjecture is true.

Of course these two assumptions are too strong. For example,

$$(A^3+2B)^2 A^{r/2}(A^3+2B)^{-2} B^{1-r}(A^3+2B)^{-2} A^{r/2}(A^3+2B)^2$$

can be considered as non-commutative analogue for $a^r b^{1-r}$. However this does not satisfy our assumptions.

We shall also consider the case $0 < r < 1$ and show a similar result. That is, under the assumptions (i) and (ii), if the inequality

$$M(A,B) \leq rA + (1-r)B$$

holds, then we will show that

$$M(A,B) = B^{1/2}(B^{-1/2} A B^{-1/2})^r B^{1/2}.$$

Our result can be considered as a characterization of r-mean, in particular the geometric mean. In the paper [3] T. Ando and K. Nishio gave a characterization of the harmonic mean.

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2. Main Result

Throughout this paper we assume that the readers are familiar with basic notations and results on operator theory. We refer the readers to Conway's book [4].

We denote by $\mathfrak{H}$ a (finite or infinite dimensional) complex Hilbert space and by $B(\mathfrak{H})$ all bounded linear operators on it. For each operator $A \in B(\mathfrak{H})$, its

operator norm is denoted by $||A||$. We denote by $B(\mathfrak{H})^+$ the set of all positive invertible operators. For two vectors $\xi, \eta \in \mathfrak{H}$, their inner product and norm are denoted by $\langle \xi, \eta \rangle$ and $||\xi||$ respectively.

In this paper we consider the map $M(\cdot, \cdot)$ from $B(\mathfrak{H})^+ \times B(\mathfrak{H})^+$ to $B(\mathfrak{H})^+$.

We fix a positive number $r > 0$. For $A, B \in B(\mathfrak{H})^+$, define

$$M_r(A, B) = B^{1/2}(B^{-1/2}AB^{-1/2})^r B^{1/2}.$$

Here we remark that

$$B^{1/2}(B^{-1/2}AB^{-1/2})^r B^{1/2} = A^{1/2}(A^{-1/2}BA^{-1/2})^{1-r}A^{1/2}. \qquad (\dagger)$$

(This is well-known for specialists.) Indeed, if r is an integer, direct computations show this equality. Thus for any polynomial $p(t)$ with $p(0) = 0$, we have

$$B^{1/2} \cdot p(B^{-1/2}AB^{-1/2}) \cdot B^{1/2} = A^{1/2} \cdot p((A^{-1/2}BA^{-1/2})^{-1}) \cdot (A^{-1/2}BA^{-1/2}) \cdot A^{1/2}.$$

Thus by continuity we get $(\dagger)$. The map M_r is so-called r-mean, and usually the case $0 < r < 1$ is considered. (When $0 < r < 1$, $M_r(A, B)$ is one of the so-called operator means. In particular, in the case $r = 1/2$, $M_r(A, B)$ is said to be the geometric mean.)

First we shall consider the case $r > 1$. The following is our main result.

Theorem 2.1. *Assume $r > 1$. For any $A, B \in B(\mathfrak{H})^+$, if the map M satisfies*

(i) $M(A, B) \geq rA + (1 - r)B$,
(ii) $M(tA, B) = t^r M(A, B)$ *for any positive number t,*
(iii) $M(A, B)^{-1} = M(A^{-1}, B^{-1})$,

then we have $M = M_r$.

We need some preparations to prove this theorem. The following lemma states that under the assumptions (i) and (ii), the map $M_r(A, B)$ is "less" than $M(A, B)$ in a certain sense. (See Remark 2.1.)

Lemma 2.2. *For any $A, B \in B(\mathfrak{H})^+$, we assume that the map M satisfies*

(i) $M(A, B) \geq rA + (1 - r)B$,
(ii) $M(tA, B) = t^r M(A, B)$ *for any positive number t.*

Then for any unit vector $\xi \in \mathfrak{H}$, if $r \geq 2$ we have

$$\langle A^{-1/2}M(A, B)A^{-1/2}\xi, \xi\rangle\langle(A^{-1/2}M_r(A, B)A^{-1/2})^{-1}\xi, \xi\rangle \geq 1.$$

On the other hand if $1 < r \leq 2$ we have

$$\langle(A^{-1/2}M(A, B)A^{-1/2})^{1/(r-1)}\xi, \xi\rangle\langle(A^{-1/2}M_r(A, B)A^{-1/2})^{-1/(r-1)}\xi, \xi\rangle \geq 1.$$

Proof. By assumptions we have

$$t^r M(A,B) \geq rtA + (1-r)B$$

and hence

$$A^{-1/2}M(A,B)A^{-1/2} \geq rt^{1-r} + (1-r)t^{-r}A^{-1/2}BA^{-1/2}.$$

For a unit vector $\xi \in \mathfrak{H}$, set

$$f(t) = rt^{1-r} + (1-r)t^{-r}\langle A^{-1/2}BA^{-1/2}\xi, \xi\rangle.$$

Here we remark that $\langle A^{-1/2}M(A,B)A^{-1/2}\xi,\xi\rangle \geq f(t)$. Then it is easy to see that the maximum value of $f(t)$ on $(0,\infty)$ is equal to $\langle A^{-1/2}BA^{-1/2}\xi,\xi\rangle^{1-r}$. Thus we get

$$\langle A^{-1/2}M(A,B)A^{-1/2}\xi,\xi\rangle\langle A^{-1/2}BA^{-1/2}\xi,\xi\rangle^{r-1} \geq 1.$$

In the case $r \geq 2$, by the Jensen inequality and (†) we have

$$\langle A^{-1/2}BA^{-1/2}\xi,\xi\rangle^{r-1} \leq \langle (A^{-1/2}BA^{-1/2})^{r-1}\xi,\xi\rangle = \langle (A^{-1/2}M_r(A,B)A^{-1/2})^{-1}\xi,\xi\rangle.$$

So we are done.

Next we consider the case $1 < r \leq 2$. Let s be a positive number such that $\frac{1}{r} + \frac{1}{s} = 1$. Then since $(r-1) = 1/(s-1)$, we have

$$\langle A^{-1/2}M(A,B)A^{-1/2}\xi,\xi\rangle\langle A^{-1/2}BA^{-1/2}\xi,\xi\rangle^{1/(s-1)} \geq 1$$

and hence

$$\langle A^{-1/2}M(A,B)A^{-1/2}\xi,\xi\rangle^{s-1}\langle A^{-1/2}BA^{-1/2}\xi,\xi\rangle \geq 1.$$

Since $s \geq 2$ and $(s-1) = 1/(r-1)$, we compute as above

$$\begin{aligned}
&\langle (A^{-1/2}M(A,B)A^{-1/2})^{1/(r-1)}\xi,\xi\rangle\langle (A^{-1/2}M_r(A,B)A^{-1/2})^{-1/(r-1)}\xi,\xi\rangle \\
&= \langle (A^{-1/2}M(A,B)A^{-1/2})^{s-1}\xi,\xi\rangle\langle A^{-1/2}BA^{-1/2}\xi,\xi\rangle \\
&\geq \langle (A^{-1/2}M(A,B)A^{-1/2})\xi,\xi\rangle^{s-1}\langle A^{-1/2}BA^{-1/2}\xi,\xi\rangle \geq 1.
\end{aligned}$$

□

Theorem 2.3. *For two positive invertible operators $X, Y \in B(\mathfrak{H})^+$, if they satisfy*

$$\langle X\xi,\xi\rangle\langle Y^{-1}\xi,\xi\rangle \geq 1$$

and

$$\langle Y\xi,\xi\rangle\langle X^{-1}\xi,\xi\rangle \geq 1$$

for any unit vector $\xi \in \mathfrak{H}$, then we have $X = Y$.

In order to show this theorem, we need the following lemma.

Lemma 2.4. *For two operators $X, Y \in B(\mathfrak{H})^+$, they satisfy*

$$\langle X\xi, \xi\rangle\langle Y^{-1}\xi, \xi\rangle \geq 1$$

for any unit vector $\xi \in \mathfrak{H}$ if and only if we have

$$tX + (tY)^{-1} \geq 2$$

for any positive number t.

Proof. Let $\xi \in \mathfrak{H}$ be a unit vector. We set $f(t) = t\langle X\xi, \xi\rangle + t^{-1}\langle Y^{-1}\xi, \xi\rangle$ for $t > 0$. Then it is easy to see that the minimum value of $f(t)$ is equal to $2\sqrt{\langle X\xi, \xi\rangle\langle Y^{-1}\xi, \xi\rangle}$. Hence we are done. □

Proof of Theorem 2.3. By the previous lemma we have

$$tX + (tY)^{-1} \geq 2$$

and

$$tY + (tX)^{-1} \geq 2$$

for any positive number t. Let $Z = Y^{1/2}XY^{1/2}$. Then we have $tZ + t^{-1} \geq 2Y$ and $t + (tZ)^{-1} \geq 2Y^{-1}$. So we get

$$\frac{2tZ}{t^2Z+1} \leq Y \leq \frac{t^2Z+1}{2t}. \tag{1}$$

First we assume that the Hilbert space is finite dimensional because in this case the proof becomes simpler. Take any projection P of rank one which reduces Z, that is, $ZP = \lambda P$ for some positive number λ. (Here we use the fact that Z is atomic, thanks to finite dimensionality.) Then we get

$$\frac{2t\lambda}{t^2\lambda+1}P \leq PYP \leq \frac{t^2\lambda+1}{2t}P.$$

Since P is of rank one, PYP is of the form $PYP = \alpha P$ for some $\alpha > 0$. Therefore taking the maximum in t on the left-hand side and the minimum on the right-hand side we have $PYP = \lambda^{1/2}P$. On the other hand, since we also have

$$\frac{t^2Z+1}{2tZ} \geq Y^{-1} \geq \frac{2t}{t^2Z+1}, \tag{2}$$

we get

$$\frac{t^2\lambda+1}{2t\lambda}P \geq PY^{-1}P \geq \frac{2t}{t^2\lambda+1}P$$

and hence $PY^{-1}P = \lambda^{-1/2}P$ as above.

Let $\xi \in \mathfrak{H}$ be a vector satisfying $P\xi = \xi$. Then we have

$$\begin{aligned}||Y^{1/2}\xi|| \cdot ||Y^{-1/2}\xi|| &= \langle PYP\xi, \xi\rangle^{1/2}\langle PY^{-1}P\xi, \xi\rangle^{1/2} = \langle \lambda^{1/2}\xi, \xi\rangle^{1/2}\langle \lambda^{-1/2}\xi, \xi\rangle^{1/2} \\ &= ||\xi||^2 = \langle Y^{1/2}\xi, Y^{-1/2}\xi\rangle.\end{aligned}$$

By the equality condition for the Cauchy-Schwarz inequality, this implies that $Y^{1/2}\xi$ is a scalar-multiple of $Y^{-1/2}\xi$, in other words, $Y\xi$ is a scalar-multiple of ξ. So we get $YP = PYP = Z^{1/2}P$. Since by the spectral theory for a positive matrix there are such projections P_i of rank one such that $\sum_i P_i = 1$, we conclude that $Y = Z^{1/2} = (Y^{1/2}XY^{1/2})^{1/2}$ and hence $X = Y$.

Next we shall consider the general case. The following argument is due to a private communication with T. Ando [2]. The author would like to thank Professor Ando for permitting the author to include his argument in this paper.

It is easy to see that for any $t > 0$

$$\frac{2tZ}{t^2Z+1} \leq Z^{1/2} \leq \frac{t^2Z+1}{2t}$$

and

$$\frac{t^2Z+1}{2tZ} \geq Z^{-1/2} \geq \frac{2t}{t^2Z+1}.$$

Combining these with (1) and (2), we have

$$\frac{2tZ}{t^2Z+1} - \frac{t^2Z+1}{2t} \leq Y - Z^{1/2} \leq \frac{t^2Z+1}{2t} - \frac{2tZ}{t^2Z+1}$$

and

$$\frac{t^2Z+1}{2tZ} - \frac{2t}{t^2Z+1} \geq Y^{-1} - Z^{-1/2} \geq \frac{2t}{t^2Z+1} - \frac{t^2Z+1}{2tZ}.$$

Here we compute

$$\frac{t^2Z+1}{2t} - \frac{2tZ}{t^2Z+1} = (t - Z^{-1/2})^2 \frac{Z^2(t+Z^{-1/2})^2}{2t(t^2Z+1)}$$

and

$$\frac{t^2Z+1}{2tZ} - \frac{2t}{t^2Z+1} = (t - Z^{-1/2})^2 \frac{Z(t+Z^{-1/2})^2}{2t(t^2Z+1)}.$$

Therefore there is a positive number γ such that for any spectrum λ of Z and a projection P which reduces Z we have

$$||PYP - Z^{1/2}P|| \leq \gamma||(\lambda - Z^{-1/2})P||^2 \tag{3}$$

and

$$||PY^{-1}P - Z^{-1/2}P|| \leq \gamma||(\lambda - Z^{-1/2})P||^2. \tag{4}$$

Let us use $(PY^{-1}P)^{-1}$ to denote the inverse of $PY^{-1}P$ on $P\mathfrak{H}$. Then we see that

$$PY^{-1}P - Z^{-1/2}P = (PY^{-1}P)(Z^{1/2}P - (PY^{-1}P)^{-1})Z^{-1/2}P$$

and hence by using (4) there is a positive number γ' such that

$$||(PY^{-1}P)^{-1} - Z^{1/2}P|| \leq \gamma'||(\lambda - Z^{-1/2})P||^2.$$

Combining this with (3) we conclude that there is a positive number γ'' such that for any spectrum λ of Z and spectral projection P of Z

$$||PYP - (PY^{-1}P)^{-1}|| \leq \gamma'' ||(\lambda - Z^{-1/2})P||^2. \tag{5}$$

For any integer n, take a partition of unity $\{P_i\}_{i=1}^n$ which consists of spectral projections of Z such that there exist spectrums $\{\lambda_i\}_{i=1}^n$ of $Z^{-1/2}$ satisfying

$$||(\lambda_i - Z^{-1/2})P_i|| \leq \frac{||Z^{-1/2}||}{n}.$$

Then it follows from (3) that

$$||\sum_{i=1}^n (P_iYP_i - Z^{1/2}P_i)|| \leq \frac{\gamma ||Z^{-1/2}||^2}{n^2}. \tag{6}$$

Similarly it follows from (5) that

$$||P_iYP_i - (P_iY^{-1}P_i)^{-1}|| \leq \frac{\gamma'' ||Z^{-1/2}||^2}{n^2}.$$

Recall the following formula, which is so-called Schur complement

$$(P_iY^{-1}P_i)^{-1} = P_iYP_i - P_iYP_i^{\perp}(P_i^{\perp}YP_i^{\perp})^{-1}P_i^{\perp}YP_i$$

where $P_i^{\perp} = 1 - P_i$. Indeed we can show

$$\begin{aligned}
\{P_iYP_i - &P_iYP_i^{\perp}(P_i^{\perp}YP_i^{\perp})^{-1}P_i^{\perp}YP_i\} \cdot P_iY^{-1}P_i \\
&= P_iYP_iY^{-1}P_i - P_iYP_i^{\perp}(P_i^{\perp}YP_i^{\perp})^{-1}P_i^{\perp}YP_iY^{-1}P_i \\
&= P_iYP_iY^{-1}P_i - P_iYP_i^{\perp}(P_i^{\perp}YP_i^{\perp})^{-1}(P_i^{\perp}Y - P_i^{\perp}YP_i^{\perp})Y^{-1}P_i \\
&= P_iYP_iY^{-1}P_i - P_iYP_i^{\perp}Y^{-1}P_i = P_i.
\end{aligned}$$

By using this formula we have

$$\begin{aligned}
||P_i^{\perp}YP_i||^2 &= ||(P_i^{\perp}YP_i^{\perp})^{1/2}(P_i^{\perp}YP_i^{\perp})^{-1/2}P_i^{\perp}YP_i||^2 \\
&\leq ||Y|| \cdot ||(P_i^{\perp}YP_i^{\perp})^{-1/2}P_i^{\perp}YP_i||^2 \\
&= ||Y|| \cdot ||P_iYP_i^{\perp}(P_i^{\perp}YP_i^{\perp})^{-1}P_i^{\perp}YP_i|| \\
&= ||Y|| \cdot ||P_iYP_i - (P_iY^{-1}P_i)^{-1}|| \\
&\leq \frac{\gamma'' ||Y|| \cdot ||Z^{-1/2}||^2}{n^2}.
\end{aligned}$$

Therefore for each unit vector $\xi \in \mathfrak{H}$ by using the Cauchy-Schwarz inequality we see that

$$\begin{aligned}
||\sum_{i=1}^{n} P_i^{\perp} Y P_i \xi|| &\leq \sum_{i=1}^{n} ||P_i^{\perp} Y P_i|| \cdot ||P_i \xi|| \\
&\leq \sqrt{\sum_{i=1}^{n} ||P_i^{\perp} Y P_i||^2} \sqrt{\sum_{i=1}^{n} ||P_i \xi||^2} \\
&= \sqrt{\sum_{i=1}^{n} ||P_i^{\perp} Y P_i||^2} \\
&\leq \sqrt{\sum_{i=1}^{n} \frac{\gamma'' ||Y|| \cdot ||Z^{-1/2}||^2}{n^2}} = \sqrt{\frac{\gamma'' ||Y|| \cdot ||Z^{-1/2}||^2}{n}}.
\end{aligned}$$

Thus we get

$$||\sum_{i=1}^{n} P_i^{\perp} Y P_i|| \leq \sqrt{\frac{\gamma'' ||Y|| \cdot ||Z^{-1/2}||^2}{n}}. \tag{7}$$

By using (6) and (7) we see that

$$\begin{aligned}
||Y - Z^{1/2}|| &\leq ||\sum_{i=1}^{n} P_i Y P_i - Z^{1/2} P_i|| + ||\sum_{i=1}^{n} P_i^{\perp} Y P_i|| \\
&\leq \frac{\gamma ||Z^{-1/2}||}{n^2} + \sqrt{\frac{\gamma'' ||Y|| \cdot ||Z^{-1/2}||^2}{n}}.
\end{aligned}$$

By tending $n \to \infty$ we get $Y = Z^{1/2}$ and hence $X = Y$.

□

Now we can prove our main result.

Proof of Theorem 2.1. First we consider the case $r \geq 2$. Set

$$X = A^{-1/2} M(A, B) A^{-1/2} \text{ and } Y = A^{-1/2} M_r(A, B) A^{-1/2}.$$

By Lemma 2.2 for any unit vector $\xi \in \mathfrak{H}$ we have

$$\langle X\xi, \xi \rangle \langle Y^{-1}\xi, \xi \rangle \geq 1.$$

On the other hand, thanks to the relations $M(A, B)^{-1} = M(A^{-1}, B^{-1})$ and $M_r(A, B)^{-1} = M_r(A^{-1}, B^{-1})$, applying Lemma 2,2 for the pair (A^{-1}, B^{-1}) we have

$$\begin{aligned}
&\langle X^{-1}\xi, \xi \rangle \langle Y\xi, \xi \rangle \\
&\quad = \langle A^{1/2} M(A^{-1}, B^{-1}) A^{1/2} \xi, \xi \rangle \langle (A^{1/2} M_r(A^{-1}, B^{-1}) A^{1/2})^{-1} \xi, \xi \rangle \geq 1.
\end{aligned}$$

Therefore by Theorem 2.3 we get $X = Y$ and hence $M = M_r$.

In the case $1 < r \leq 2$, set

$$X = (A^{-1/2}M(A,B)A^{-1/2})^{1/(r-1)} \text{ and } Y = (A^{-1/2}M_r(A,B)A^{-1/2})^{1/(r-1)}.$$

Then in the same way we conclude the desired fact. $\square$

Remark 2.1. (i) For positive invertible operators A, B, C, the block matrix

$$\begin{pmatrix} A & B \\ B & C \end{pmatrix}$$

is positive if and only if $A \geq BC^{-1}B$ [1]. Therefore for two positive invertible operators X, Y, the block matrix

$$\begin{pmatrix} X & 1 \\ 1 & Y^{-1} \end{pmatrix}$$

is positive if and only if $X \geq Y$. On the other hand for any unit vector $\xi \in \mathfrak{H}$ the matrix

$$\begin{pmatrix} \langle X\xi, \xi\rangle & 1 \\ 1 & \langle Y^{-1}\xi, \xi\rangle \end{pmatrix}$$

is positive if and only if $\langle X\xi, \xi\rangle\langle Y^{-1}\xi, \xi\rangle \geq 1$. Thus the condition $\langle X\xi, \xi\rangle\langle Y^{-1}\xi, \xi\rangle \geq 1$ is weaker than $X \geq Y$. We do not know whether the condition $\langle X\xi, \xi\rangle\langle Y^{-1}\xi, \xi\rangle \geq 1$ define new order X" $\geq$" Y or not. The author guess that this relation does not satisfy transitivity. Here we remark that if X" $\geq$" Y, then we have X^2" $\geq$" Y^2. Indeed if we have $\langle X\xi, \xi\rangle\langle Y^{-1}\xi, \xi\rangle \geq 1$, then we get

$$\langle X^2\xi, \xi\rangle\langle Y^{-2}\xi, \xi\rangle \geq \langle X\xi, \xi\rangle^2\langle Y^{-1}\xi, \xi\rangle^2 \geq 1.$$

Thus this relation is not equivalent to usual order. Theorem 2.3 states that if we have X" $\geq$" Y and Y" $\geq$" X, then we conclude $X = Y$ (reflexivity).

(ii) We would like to conjecture that Theorem 2.1 holds by replacing the condition (iii) with

$$\text{(iii)}' \quad M(A,B) = A^rB^{1-r} \quad \text{if } A \text{ commutes with } B.$$

Finally we shall prove the analogue in the case $0 < r < 1$ for Theorem 2.1.

Theorem 2.5. *Assume $0 < r < 1$. For any $A, B \in B(\mathfrak{H})^+$, if the map M satisfies*

(i) $M(A,B) \leq rA + (1-r)B$,
(ii) $M(tA,B) = t^rM(A,B)$ *for any positive number* t,
(iii) $M(A,B)^{-1} = M(A^{-1}, B^{-1})$,

then we have $M = M_r$.

Proof. The proof is essentially same as that of Theorem 2.1. So we would like to give the sketch of the proof.

By assumptions for any positive number t we have

$$M(A,B) \leq rt^{r-1}A + (1-r)t^r B$$

and

$$M(A,B)^{-1} \leq rt^{1-r}A^{-1} + (1-r)t^{-r}B^{-1}.$$

Set

$$Y = B^{-1/2}M(A,B)B^{-1/2} \text{ and } Z = B^{-1/2}AB^{-1/2}.$$

Then we have

$$\frac{t^r Z}{rt + (1-r)Z} \leq Y \leq \frac{rZ + (1-r)t}{t^{1-r}}.$$

Then by the almost same arguments as those in the proof of Theorem 2.3, we can show $Y = Z^r$. □

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